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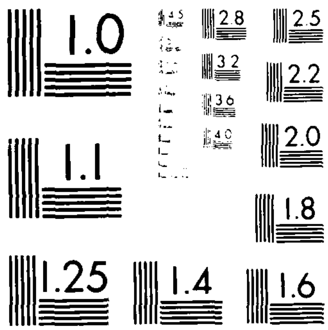

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ADMISSIBLE AND SIGNULAR TRANSLATES OF STABLE PROCESSES

by

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Technical Report No. 201

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ABSTRACT

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Keywords and Phrases: Stable processes, admissible translates, equivalence and singularity, harmonizable processes, moving averages, Gaussian mixtures.

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0. Introduction and Summary

The Lebesgue decomposition of measures induced by stochastic processes is important in areas such as statistical inference and information theory. Of particular interest is the Lebesgue decomposition between the measures induced by a stochastic process and its translate by a nonrandom function, i.e., the problem of detecting a nonrandom signal in additive random noise.

For Gaussian processes the Lebesgue decomposition has been fully described and the following dichotomy prevails: two Gaussian processes are either mutually absolutely continuous, and then their discrimination is based on a threshold test on the log of their Radon-Nikodym derivative (log likelihood ratio) which has a known expression, or else they are singular, and then they can be discriminated with probability one (see e.g. [9]). Some partial results are also available for processes having finite second moments [11].

The Central Limit Theorem and the stability property provide the basic reasons for regarding stable processes as a natural generalization of Gaussian processes. Most of the work on stable processes focuses on contrasts and similarities between Gaussian and non-Gaussian stable processes. While the detection of a nonrandom signal in additive Gaussian noise has been thoroughly studied, the problem of detecting a nonrandom signal in additive stable noise has remained largely open.

This work investigates the equivalence and singularity of measures induced by non-Gaussian stable processes and their translates. For non-Gaussian measures, these questions seem to have been first studied in [12] for infinitely divisible measures in Hilbert space and subsequently in [24] and [22] for stable measures.

Sufficient conditions for an element to be an admissible translate of an infinitely divisible measure in a Hilbert space were obtained in [12]. However, as observed in [24], these conditions are difficult to verify and, as simplified for stable measures, they were found to be false. The structure of the set of all admissible translates of symmetric stable measures was investigated in [24], where it was shown that certain stable processes have no admissible translate. The admissible translates of a symmetric stable measure with discrete spectral measure were characterized in [22].

All these works use primarily the representation of the characteristic functional of stable measures in Hilbert or Banach space. Here we work with stable processes and exploit their spectral representation, which in some cases allows the formulation of the problem in terms of processes with independent increments or sequences of independent random variables.

The first section of this paper introduces the setting and notation, and presents the basic definitions and results on stable processes.

For p^{th}-order and for symmetric stable processes a function space is introduced in Section 2 which plays a role partly analogous to the reproducing kernel Hilbert space of a

Gaussian or second order process. In particular this space provides an upper bound for the set of admissible translates, is a stochastic processes version of a space introduced in [24, p. 249], and extends the results of [24, Proposition 10] to general symmetric stable processes and of [11, Theoreme 4.1] to general p^{th}-order processes. A lower bound for the set of admissible translates of a stable process is also provided by exploiting their structures as mixtures of Gaussian processes, and a dichotomy is shown for a class of stable processes which includes all sub-Gaussian and sub-Gaussian-like processes.

In Section 3 stable processes with an invertible spectral representation are considered. Their admissible translates are characterized, and a dichotomy is established: each translate is either admissible or singular. The result is applied to show that most continuous time moving averages, and all harmonizable processes with nonatomic spectral measure have no admissible translate. Thus these processes do not provide realistic models for additive noise, as every nonrandom signal can be perfectly detected in their presence. For general harmonizable and for invertible discrete time mixed autoregressive moving average processes the set of admissible translates is characterized.

1. Background and notation

The following setting is considered. $X = (X(t,\omega) = X(t); t \in \mathbb{T})$ is a stochastic process on a probability space (Ω, F, P) with parameter set $\mathbb{T}$ and real or complex values, i.e. values in $X = \mathbb{R}$ or $\mathbb{C}$. When $X(t) \in L_p(\Omega, F, P) = L_p(P)$ for all $t \in \mathbb{T}$, where $p > 0$, X is called a p^{th} order process. The linear space $L(X)$ of a p^{th} order process X is the $L_p(P)$ completion of the set of finite linear combinations of its random variables $\ell(X) \triangleq sp\{X(t); t \in \mathbb{T}\}$. $\overline{X}^{\mathbb{T}}$ denotes the set of all extended X (i.e., real or complex) valued functions on $\mathbb{T}$, $C = C(\overline{X}^{\mathbb{T}})$ the σ-field generated by the cylinder sets of $\overline{X}^{\mathbb{T}}$ and μ_X the distribution of the process X, i.e. the probability induced on C by X:

$$\mu_X(C) = P(\{\omega; X(\cdot,\omega) \in C\}),\ C \in C.$$

For a nonrandom real or complex function s on $\mathbb{T}$, we are interested in the Lebesgue decomposition of the distribution μ_{s+X} of the process $s + X$ with respect to μ_X; and in particular in conditions for μ_{s+X} and μ_X to be singular ($\mu_{s+X} \perp \mu_X$), and for μ_{s+X} to be absolutely continuous with respect to μ_X ($\mu_{s+X} << \mu_X$). The function s is then called a singular or admissible translate of X respectively.

Here we focus on symmetric α-stable (SαS) processes. A real random variable X is SαS, $0 < \alpha \le 2$, with scale parameter $||X||_\alpha \in (0,\infty)$ if $E\{\exp(iuX)\} = \exp\{-||X||_\alpha^\alpha |u|^\alpha\}$. A real random vector $(X_1, \ldots, X_n)$ is SαS (or its components are jointly SαS) if all linear combinations $\Sigma_{k=1}^n a_k X_k$ are SαS. Similarly a real stochastic process $X = (X(t); t \in \mathbb{T})$ is SαS if all linear combinations $\Sigma_{k=1}^n a_k X(t_k)$ are SαS random variables. When $\alpha = 2$ we have zero mean Gaussian random variables, vectors and processes respectively. When $0 < \alpha < 2$, the tails of the distributions are heavier and only moments of order $p \in (0,\alpha)$ are finite with $\{E(|X|^p)\}^{1/p} = C_{p,\alpha}||X||_\alpha$, where the constant $C_{p,\alpha}$ is independent of X. Thus a SαS process X is p^{th} order for all $0 < p < \alpha$, and its linear space $L(X)$ does not depend on p and is the completion of $\ell(X)$ with respect to $||\cdot||_\alpha^{1\wedge\alpha}$, which in fact metrizes convergence in probability ([19]).

An important class of SαS processes consists of SαS independently scattered random measures, which extend the concept of a stochastic process with independent increments to more general parameter spaces. Let $\mathbb{I}$ be an arbitrary set and I a δ-ring of subsets of $\mathbb{I}$ with the property that there exists an increasing sequence $(I_n; n \in \mathbb{N})$ in I with $\cup_n I_n = \mathbb{I}$. A real stochastic process $Z = (Z(A); A \in I)$ is called an independently scattered SαS random measure if for every sequence $(A_n;\ n \in \mathbb{N})$ of disjoint sets in I, the random variables $Z(A_n)$, $n = 1,2,\ldots$ are independent, and whenever $\cup_n A_n \in I$ then $Z(\cup_n A_n) = \Sigma_n Z(A_n)$ a.s., and for every $A \in I$, $Z(A)$ is a SαS random variable, i.e. $E\{\exp(iuZ(A)\} = \exp\{-m(A)|u|^\alpha\}$ where $m(A) = ||Z(A)||_\alpha^\alpha$. Then m is a measure on I which extends uniquely to a σ-finite measure on $\sigma(I)$, and is called the control measure of Z. Conversely, the existence of an independently scattered SαS random measure with a given control measure is a consequence of Kolmogorov's consistency theorem.

When $\mathbb{I}$ is an interval of the real line, there is an identification between independent increments processes and independently scattered random measures. Namely if $X = (X(t), t \in \mathbb{I})$ is an independent increments process and $(a,b] \subset \mathbb{I}$: an interval,

$Z((a,b]) \triangleq X(b) - X(a)$ can be extended to an independently scattered random measure on the δ-ring $\mathcal{I}$ of bounded Borel sets of $\mathbb{I}$. Conversely given an independently scattered random measure Z on $\mathcal{I}$, and a in $\mathbb{I}$, $X(t) = \mathrm{sign}(t-a)Z((a \wedge t, a \vee t])$, $t \in \mathbb{I}$, is an independent increments process. When the control measure m is Lebesgue measure, then X has stationary independent increments, $E\{\exp(iu[X(t)-X(t)])\} = \exp\{-|t-t'|\,|u|^{\alpha}\}$, and is called S$\alpha$S motion on $\mathbb{I}$.

For any function $f \in L_\alpha(\mathbb{I}, \sigma(\mathcal{I}), m) = L_\alpha(m)$ the stochastic integral $\int_{\mathbb{I}} f dZ$ can be defined in the usual way and is a SαS random variable with $||\int f dZ||_\alpha = ||f||_{L_\alpha(m)}$. The stochastic integral map $f \to \int_{\mathbb{I}} f dZ$ from $L_\alpha(m)$ into $L(Z)$ is an isometry and

(1.1) $\quad L(Z) = \{\int f dZ; f \in L_\alpha(m)\}.$

The stochastic integral allows for the construction of SαS processes with generally dependent values by means of the spectral representation

(1.2) $$X(t) = \int_{\mathbb{I}} f(t,u)Z(du), t \in \mathbb{T},$$

where $\{f(t,\cdot); t \in \mathbb{T}\} \subset L_\alpha(m)$. In fact every S$\alpha$S process X has such a spectral representation in law, in the sense that for some family $\{f(t,\cdot), t \in \mathbb{T}\}$ in some $L_\alpha(m)$,

(1.3) $$(X(t); t \in \mathbb{T}) \stackrel{L}{=} (\int_{\mathbb{I}} f(t,u)Z(du); t \in \mathbb{T})$$

(see e.g., [16] and [13]). If $L(X)$ is separable, e.g., X is continuous in probability, then $L_\alpha(\mathbb{I}, m)$ can be chosen as $L_\alpha([0,1], \mathrm{Leb})$.

Specific examples of SαS processes will be considered in the following sections.

The covariation $[X,Y]_\alpha$ of two jointly SαS random variables X and Y with $1 < \alpha \le 2$ is defined by

$$(1.4) \qquad \frac{[X,Y]_\alpha}{||Y||_\alpha^\alpha} = \frac{E(XY^{<p-1>})}{E(|Y|^p)}$$

which holds for all $0 < p < \alpha$ (where $y^{<q>} = |y|^{q-1}y$, $q > 0$) (see e.g. [6]). It follows that $||X||_\alpha^\alpha = [X,X]_\alpha$. If X and Y have representations $\int_{\mathrm{II}} f dZ$ and $\int_{\mathrm{II}} g dZ$ respectively then $[X,Y]_\alpha = \int_{\mathrm{II}} f g^{<\alpha-1>} dm$.

In certain cases, such as when working with Fourier transforms, it is more natural and convenient to work with complex valued processes. A complex SαS random variable is defined as having jointly SαS real and imaginary parts. Except for the representation of the characteristic function, all concepts and results considered in this section for real SαS random variables and processes extend to the complex case (see e.g. [5] and [6]).

2. An upper bound for the set of admissible translates

A space of functions associated with a p^{th} order, $0 < p \leq 2$, stochastic process will be introduced and seen as a partial extension of the reproducing kernel Hilbert space (RKHS) associated with a second order process. We concentrate only on p^{th} order processes with $p < 2$ because for those with $p \geq 2$ the second order theory is applicable.

Recall that for a second order stochastic process $X = (X(t); t \in \mathbb{T})$ with arbitrary index set $\mathbb{T}$, zero mean and covariance function R, the RKHS H of X (or of R) consists of all functions s of the form $s(t) = E(X(t)\overline{Y})$, $t \in \mathbb{T}$, $Y \in L(X)$. If $s_i(t) = E(X(t)\overline{Y}_i)$ then $\langle s_1, s_2 \rangle_H = E(Y_1 \overline{Y}_2)$ defines an inner product and R is a reproducing kernel, i.e. for all $t \in \mathbb{T}$, $R(\cdot, t) \in H$ and $s(t) = \langle s, R(\cdot, t) \rangle_H$. Also $s \in H$ if and only if

$$||s||_H = \sup \frac{\left| \sum_{n=1}^{N} a_n s(t_n) \right|}{\left[E \left| \sum_{n=1}^{N} a_n X(t_n) \right|^2\right]^{1/2}} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$, $a_1, \ldots, a_N \in X$ and $t_1, \ldots, t_N \in \mathbb{T}$.

We now introduce the function space of a p^{th} order process with $0 < p \leq 2$ and arbitrary index $\mathbb{T}$, and present its properties.

Definition 2.1. The function space $\mathbb{F} = \mathbb{F}(X)$ of a p^{th} order process $X = (X(t); t \in \mathbb{T})$ with $0 < p \leq 2$ is the set of all functions s on $\mathbb{T}$ such that

$$||s||_{\mathbb{F}} \triangleq \sup \frac{\left| \sum_{n=1}^{N} a_n s(t_n) \right|}{\left[E \left| \sum_{n=1}^{N} a_n X(t_n) \right|^p\right]^{1/p}} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$, $a_1, \ldots, a_N \in X$ and $t_1, \ldots, t_N \in \mathbb{T}$.

When $1 < p \leq 2$, a representation is known for the bounded linear functionals on the linear space of X, analogous to the Riesz representation for bounded linear functionals on a Hilbert space. This allows us to express the functions in $\mathbb{F}$ in terms of moments of the process X. This and further properties of the function space are collected in the following

Proposition 2.2. *Let* $X = (X(t); t \in \mathbb{T})$ *be a* p^{th} *order process with* $1 < p \leq 2$. *Then the following three statements are equivalent:*

i) $s \in \mathbb{F}$,

ii) $s(t) = E(X(t)Y^{<p-1>})$ *for* $Y \in L(X)$,

iii) $s(t) = E(X(t)\overline{W})$ *for* $W \in L_{p^*}(P)$ *where* $1/p + 1/p^* = 1$.

Moreover the following properties hold.

a) $||s||_{\mathbb{F}} = ||Y||^{p-1}_{L_p(P)}$ *if* $s(t) = E(X(t)Y^{<p-1>})$, $Y \in L(X)$.

b) For each $s \in \mathbb{F}$, *with* $s(t) = E(X(t)Y^{<p-1>})$, $Y \in L(X)$, *there exists a unique* $W \in L_{p^*}(P)$ *(namely* $\overline{W} = Y^{<p-1>}$*) satisfying iii and* $||s||_F = ||W||_{L_{p^*}(P)}$.

c) $(\mathbb{F}, || \ ||_{\mathbb{F}})$ *is a Banach space isometrically isomorphic to the quotient space* $L_{p^*}(P)/\ell(X)^{\perp}$, *where* $\ell(X)^{\perp}$ *denotes the annihilator of* $\ell(X)$.

<u>Proof</u>: <u>i $\Rightarrow$ ii</u> follows by observing that if $||s||_{\mathbb{F}} < \infty$, then $\psi_s(\Sigma_{n=1}^N a_n X(t_n)) = \Sigma_{n=1}^N a_n s(t_n)$ defines a bounded linear functional on $L(X)$ with norm $||s||_{\mathbb{F}}$. From [7, Proposition 2.1], there exists a unique $Y \in L(X)$ such that $\psi_s(\cdot) = E(\cdot Y^{<p-1>})$ and

$\|\phi_s\|_{L(X)^*} = \|Y\|^{p-1}_{L_p(P)}$. Thus $s(t) = \phi_s(X(t)) = E(X(t)Y^{<p-1>})$.

ii $\Rightarrow$ iii. If $Y \in L(X)$ then $\bar{W} = Y^{<p-1>} \in L_{p*}(P)$ and $\|W\|_{L_{p*}(P)} = \|Y\|^{p-1}_{L_p(P)}$. Also $s(t) = E(X(t)Y^{<p-1>}) = E(X(t)\bar{W})$.

iii $\Rightarrow$ i. If $s(t) = E(X(t)\bar{W})$ then it is clear from its definition that $\|s\|_{\mathbb{F}}$ is finite.

a. That $\|s\|_{\mathbb{F}} = \|Y\|^{p-1}_{L_p(P)}$ follows as in the proof of i $\Rightarrow$ ii.

b. Let $s \in \mathbb{F}$. By iii there exist $Z \in L_{p*}(P)$ such that $s(t) = E(X(t)\bar{Z})$. Let $\ell(X)^{\perp}$ be the closed linear space $\{Z' \in L_{p*}(\Omega); E(Z'\bar{Y}) = 0, Y \in \ell(X)\}$ and let Z_0 be the best approximation of Z in $\ell(X)^{\perp}$ i.e.

$$\|Z - Z_0\|_{L_{p*}(P)} = \inf\{\|Z - Z'\|_{L_{p*}(P)}; Z' \in \ell(X)^{\perp}\}.$$

Such a $Z_0 \in \ell(X)^{\perp}$ exist and is unique [21, Corollary 3.5 and Theorem 1.11]. Set $W = Z - Z_0$. Then $E(Z\bar{Y}) = E(W\bar{Y})$ for all $Y \in \ell(X)$ if Z' is such that $E(Z'\bar{Y}) = E(Z\bar{Y})$ for all $Y \in \ell(X)$. Then $Z - Z' \in \ell(X)^{\perp}$ and

$$\|W\|_{L_{p*}(P)} = \|Z - Z_0\|_{L_{p*}(P)} \le \|Z - (Z - Z')\|_{L_{p*}(P)} = \|Z'\|_{L_{p*}(P)}.$$

Thus if $s(t) = E(X(t)\bar{W}')$, $W' \in L_{p*}(P)$, and $\|s\|_{\mathbb{F}} = \|W'\|_{L_{p*}(P)}$ we must have $\|W\|_{L_{p*}(P)} \le \|W'\|_{L_{p*}(P)}$. On the other hand

$$\|W'\|_{L_{p*}(P)} = \|S\|_{\mathbb{F}} = \sup \frac{|E(\sum_{n=1}^{N} a_n X(t_n) W)|}{\|\sum_{n=1}^{N} a_n X(t_n)\|_{L_p(P)}} \le \|W\|_{L_{p*}(P)}.$$

Therefore $\|W\|_{L_{p*}(P)} = \|W'\|_{L_{p*}(P)} = \|s\|_{\mathbb{F}}$. Putting $V = Z - W'$ we have $V \in \ell(X)^{\perp}$ and

$$\|Z - Z_0\|_{L_{p*}(P)} = \|W\|_{L_{p*}(P)} = \|W'\|_{L_{p*}(P)} = \|Z - V\|_{L_{p*}(P)}.$$

Thus the unicity of Z_0 implies $W = W'$. Since $Y^{<p-1>} \in L_{p*}(P)$ and $\|s\|_{\mathbb{F}} = \|Y^{<p-1>}\|_{L_{p*}(P)}$ for $s(t) = E(X(t)Y^{<p-1>})$ we must have $W = Y^{<p-1>}$.

c. That $(\mathbb{F}, \|\ \|_{\mathbb{F}})$ is a normed linear space is clear. To show that $\mathbb{F}$ is isometrically isomorphic to $L_{p*}(P)/\ell(X)^{\perp}$, let $s_i \in \mathbb{F}$, $i = 1,2$,

$$s_i(t) = E(X(t)\overline{W}_i),$$

$$(s_1 + s_2)(t) = E(X(t)\overline{W}),$$

where W_1, W_2 and W are the unique elements in $L_{p*}(P)$ such that $\|s_i\|_{\mathbb{F}} = \|W_i\|_{L_{p*}(P)}$. Since $E(X(t)\overline{W}) = (s_1 + s_2)(t) = E(X(t)(\overline{W_1 + W_2}))$ we have $W - (W_1 + W_2) \in \ell(X)^{\perp}$ i.e. $[W] = [W_1 + W_2] = [W_1 + W_2]$, where $[\cdot]$ denotes an equivalence class in $L_{p*}(P)/\ell(X)$. Similarly if $s(t) = E(X(t)\overline{W})$ and $(as)(t) = E(X(t)\overline{\tilde{W}})$ we have $[\tilde{W}] = [aW] = a[W]$. Hence the map $s \to [W]$ is linear and since $\|[W]\|_{L_{p*}(P)/\ell(X)^{\perp}} = \|W\|_{L_{p*}(P)} = \|s\|_{\mathbb{F}}$ it is an isometric isomorphism.

To finish the proof of c we need to show that $\mathbb{F}$ is complete. Let $(s_k; k \in \mathbb{N})$ be a sequence in $\mathbb{F}$ such that $\Sigma_{k=1}^{\infty}\|s_k\|_{\mathbb{F}} < \infty$ and let $W_k \in L_{p*}(P)$ be such that $\|W_k\|_{L_{p*}(P)} = \|s_k\|_{\mathbb{F}}$. Hence $\Sigma_{k=1}^{\infty}\|W_k\|_{L_{p*}(P)} < \infty$ and $W = \Sigma_{k=1}^{\infty}W_k \in L_{p*}(P)$. Set $s(t) = E(X(t)\overline{W})$. Thus

$$\left|\sum_{n=1}^{N} a_n\left(\sum_{k=1}^{K} s_k - s\right)(t_n)\right| \le \left\|\sum_{k=1}^{K} W_k - W\right\|_{L_{p^*}(P)} \left\|\sum_{n=1}^{N} a_n X(t_n)\right\|_{L_p(P)}$$

and

$$\left\|\sum_{k=1}^{K} s_k - s\right\|_{\mathbb{F}} \le \left\|\sum_{k=1}^{K} W_k - W\right\|_{L_{p^*}(P)} \to 0 \quad \text{as } K \to \infty,$$

i.e. $\Sigma_{n=1}^{\infty} s_n = s \in \mathbb{F}$ proving that $\mathbb{F}$ is complete. □

Further properties of the function space $\mathbb{F}$ of the process X, for $1 < p \le 2$, analogous to those of a RKHS are the following:

i) If $\mathbb{T}$ is a metric space, functions in $\mathbb{F}$ are as "smooth" as the process X is in the weak sense, i.e., they are continuous (differentiable) if and only if X is weakly continuous (differentiable).

ii) Norm convergence in $\mathbb{F}$ implies pointwise convergence, and the convergence is uniform if $\|X(t)\|_{L_p(P)}$ is uniformly bounded

If the process X is SαS with $1 < \alpha < 2$, then it is of p^{th} order for each $p \in (1,\alpha)$ and its function space $\mathbb{F}$ does not depend on p but only on α and can be defined by means of the norm $\|\cdot\|_{\alpha}$ and described by means of covariation instead of moments. Furthermore the functions in $\mathbb{F}$ can be expressed in terms of the spectral representation of the process.

Corollary 2.3. *Let* $X = (X(t); t \in \mathbb{T})$ *be a* SαS *process with* $1 < \alpha \le 2$ *and spectral representation*

$$X(t) = \int_{\mathbb{U}} f(t,u) Z(du), \quad t \in \mathbb{T},$$

where Z *has control measure* m. *Then the following three statements are equivalent:*

i) $s \in \mathbb{F}$,

ii) $s(t) = [X(t), Y]_\alpha$ *for* $Y \in L(X)$,

iii) $s(t) = \int_{\mathbb{T}} f(t,u)\overline{z}(u)m(du)$ *for* $z \in L_{\alpha^*}(m)$ *where* $1/\alpha^* + 1/\alpha = 1$.

Moreover the following properties hold.

a) $$\|s\|_{\mathbb{F}} = C_{p,\alpha} \sup \frac{\left|\sum_{n=1}^{N} a_n s(t_n)\right|}{\left\|\sum_{n=1}^{N} a_n X(t_n)\right\|_\alpha} = \|Y\|_\alpha^{\alpha-1}.$$

b) For each $s \in \mathbb{F}$ *there exists a unique* $z \in L_{\alpha^*}(m)$ *satisfying iii and* $\|s\|_{\mathbb{F}} = \|z\|_{L_{\alpha^*}(m)}$.

c) The map $s \to [z]$ *from* $\mathbb{F}$ *into* $L_{\alpha^*}(m)/\ell(f)^\perp$, *where* $[\cdot]$ *is an equivalence class in* $L_{\alpha^*}(m)/\ell(f)^\perp$, $\ell(f) = sp\{f(t,\cdot); t \in \mathbb{T}\}$, *is an isometric isomorphism.*

<u>Proof:</u> <u>i $\Leftrightarrow$ ii</u>. It follows from (1.3) that for all $p \in (1,\alpha)$

$$[\cdot, Y]_\alpha = E(\cdot Z^{<p-1>}),$$

where $Z = C_{p,\alpha}^{-p/(p-1)} \|Y\|_\alpha^{(\alpha-p)/(p-1)} Y$ and $\|Z\|_{L_p(P)}^{p-1} = C_{p,\alpha}^{-1} \|Y\|_\alpha^{\alpha-1}$, so that $s \in \mathbb{F}$ if and only if $s(t) = [X(t), Y]_\alpha$ which does not depend on p.

<u>ii $\Rightarrow$ iii</u>. If $Y \in L(X)$ then $Y = \int_{\mathbb{T}} g\,dZ$ for some $g \in L(f) = \overline{sp}\{f(t,\cdot); t \in \mathbb{T}\}$, and

$$s(t) = [X(t), Y]_\alpha = \left[\int_{\mathbb{T}} f(t,\cdot)dZ, \int_{\mathbb{T}} g\,dZ\right]_\alpha = \int_{\mathbb{T}} f(t,\cdot) g^{<\alpha-1>} dm = \int_{\mathbb{T}} f(t,\cdot) z\,dm$$

where $z = g^{<\alpha-1>} \in L_{\alpha*}(m)$ and $\|z\|_{L_{\alpha*}(m)} = \|Y\|_{\alpha}^{\alpha-1}$.

The proofs of iii $\Rightarrow$ i, the uniqueness of z and of the isometric isomorphism are identical to those of Proposition 2.1. $\square$

Further, the well known dichotomy on the admissible translates of a Gaussian process—namely that the admissible translates of a Gaussian process are precisely the functions in its RKHS, and its translates by functions outside its RKHS are singular—has a partial analog for p^{th} order processes $0 < p \le 2$, where the RKHS is replaced by the function space $\mathbb{F}$. Our result extends that of Théorème 4.1 in [11] to p^{th} order processes $0 < p < 2$, and when applied to SαS processes with $0 < \alpha < 2$, it generalizes Proposition 10 in [24] (to any SαS process).

Proposition 2.4. Let $X = (X(t); t \in \mathbb{T})$ *be a* p^{th} *order process with* $0 < p \le 2$. *If* $s \notin \mathbb{F}$ *then* $\mu_{s+X} \perp \mu_X$. *Consequently all admissible translates of* X *belong to* $\mathbb{F}$.

Proof: The proof is adapted from [18]. If $s \notin \mathbb{F}$, then

$$\sup \frac{\left|\sum_{n=1}^{N} a_n s(t_n)\right|}{\left\|\sum_{n=1}^{N} a_n X(t_n)\right\|_{L_p(P)}} = \infty .$$

Hence for each $n \in \mathbb{N}$, we can choose N_n, $a_{n,k}$, $t_{n,k}$, $k = 1, \ldots, N_n$ such that

$$\frac{\left|\sum_{k=1}^{N_n} a_{n,k} s(t_{n,k})\right|}{\left\|\sum_{k=1}^{N_n} a_{n,k} X(t_{n,k})\right\|_{L_p(P)}} \ge n^{1/p}.$$

Let $s_n = \Sigma_{k=1}^{N_n} a_{n,k} s(t_{n,k})$, without loss of generality we can consider $s_n > 0$ for all n. Consider the random variables defined on $(\bar{X}^{\mathbb{T}}, \mathcal{C}, \mu_X)$ by

$$Y_n(x) = \sum_{k=1}^{N_n} a_{n,k} x(t_{n,k}), \quad x \in X^{\mathbb{T}}.$$

By the Markov inequality we have

$$\mu_X(Y_n \geq s_n/2) \leq \mu_X(|Y_n| \geq s_n/2) \leq 2^p \| Y_n \|^p_{L_p(P)} / s_n^p < 2^p/n \to 0 \quad \text{as } n \to \infty$$

and

$$\begin{aligned} \mu_{s+X}(Y_n \geq s_n/2) &= \mu_X(Y_n + s_n \geq s_n/2) \\ &= \mu_X(Y_n \geq -s_n/2) \geq \mu_X(|Y_n| < s_n/2) \\ &= 1 - \mu_X(|Y_n| \geq s_n/2) \geq 1 - 2^p \| Y_n \|^p_{L_p(P)} / s_n^p \\ &\geq 1 - 2^p/n \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore $\mu_X \perp \mu_{s+X}$. □

From now on we restrict our attention to SαS processes. In contrast with the Gaussian case, $\alpha = 2$, where the set of admissible translates is always the entire space $\mathbb{F}$, i.e. the RKHS, the set of admissible translates of a SαS process with $\alpha < 2$ may be as large as the entire function space $\mathbb{F}$ on as small as $\{0\}$, as is seen by the following examples.

<u>Stable Motion</u>: If $X = (X(t); t \in [0,1])$ is an SαS motion, i.e., X has stationary independent SαS increments, it is known [3 ,12 ,24] that X has no nontrivial admissible translates for $0 < \alpha < 2$. On the other

hand for $1 < \alpha < 2$, its function space is the space of absolutely continuous functions with $s(0) = 0$ and derivative in $L_{\alpha*}(\text{Leb})$, i.e.

$$\mathbb{F} = \{s; s(t) = \int_0^t s'(u)du,\ t \in [0,1],\ s' \in L_{\alpha*}(\text{Leb})\}$$

with $||s||_{\mathbb{F}} = ||s'||_{L_{\alpha*}(\text{Leb})}$.

<u>Sub-Gaussian processes</u>: Let $X = (X(t); t \in \mathbb{T})$ be an α-sub-Gaussian process, i.e. its finite dimensional characteristic functions have the form

$$E\{\exp(i \sum_{n=1}^{N} a_n X(t_n))\} = \exp\{-(\frac{1}{2} \sum_{n,m=1}^{N} a_n R(t_n, t_m) a_m)^{\alpha/2}\}$$

where R is a covariance function, or equivalently

$$(X(t); t \in \mathbb{T}) \stackrel{L}{=} (A^{1/2} G(t); t \in \mathbb{T})$$

where A is a normalized positive $\alpha/2$-stable random variable independent of the Gaussian process $G = (G(t); t \in \mathbb{T})$ which has zero mean and covariance function R. It follows from [14] that the set of admissible translates of X coincides with the RKHS of G, once we observe that there the proof depends only on the representation of spherically invariant processes as scale mixtures of Gaussian processes and not on the existence of second moments. Moreover for any $Y \in L(X)$, $[X(t), Y]_\alpha = 2^{\alpha/2}\{E(W^2)\}^{1-\alpha/2} E(G(t)W)$, where $W \in L(G)$ is obtained from G by the same linear operation Y is obtained from X (see [7]). Therefore the function space $\mathbb{F}$ of X coincides with the RKHS of G (and is therefore a Hilbert space).

Stable processes as mixtures of Gaussian processes: It has been shown [17] that every SαS process X is conditionally Gaussian with zero mean, i.e. there exists a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ such that given $\mathcal{G}$, the law of X is Gaussian with mean zero and covariance function R. Denoting by G_R such a Gaussian process and by μ_{G_R} its law, we have that for every SαS process X there exists a probability λ on the space $\mathcal{R}$ of all covariance functions R such that

$$\mu_X(E) = \int_{\mathcal{R}} \mu_{G_R}(E)\lambda(dR)$$

for all $E \in \mathcal{C}$. The SαS process X is thus a Gaussian process with random covariance function R, and it is easily checked that all quadratic forms $\Sigma_{n,m=1}^{N} a_n R(t_n, t_m) \bar{a}_m$ are positive $\alpha/2$-stable random variables. Likewise we have for all $E \in \mathcal{C}$,

$$\mu_{s+X}(E) = \int_{\mathcal{R}} \mu_{s+G_R}(E)\lambda(dR).$$

It follows that if s is an admissible translate of almost all G_R's, then it is an admissible translate of X too. This gives a lower bound for the set of admissible translates of X, namely

$$\bigcup_{\substack{\Lambda \subset \mathcal{R} \\ \lambda(\Lambda)=0}} \bigcap_{R \in \mathcal{R}\setminus\Lambda} \mathrm{RKHS}(R).$$

Thus a SαS process will have admissible translates if it is a mixture of Gaussian processes whose RKHS's have a common part, i.e. if $\cap_{R \in \mathcal{R}\setminus\Lambda} \mathrm{RKHS}(R) \neq \{0\}$ for some $\lambda(\Lambda) = 0$.

The converse does not seem to be necessarily true, i.e. an admissible translate of X may not be an admissible translate of almost all the Gaussian processes whose mixture is X.

It also follows that a singular translate of X is a singular translate of almost all the Gaussian processes whose mixture is X, and furthermore the same event separates them. This gives an upper bound for the set of singular translates of X, namely

$$\bigcup_{\substack{\Lambda \subset R \\ \lambda(\Lambda)=0}} \bigcap_{R \in R\setminus\Lambda} \mathrm{RKHS}(R)^c .$$

Conversely, if s is a singular translate of a.e. G_R (λ), it may not be a singular translate of X; but if furthermore the separating set of μ_{s+G_R} and μ_{G_R} does not depend on R a.e. (λ), then s is a singular translate of X.

When a SαS process is a mixture of Gaussian processes having the same RKHS then we show that a dichotomy prevails, with every translate being either admissible or singular.

<u>*Proposition 2.5*</u>. *Let the* SαS *process* $X = (X(t); t \in \mathbb{T})$ *be the* λ*-mixture of Gaussian processes* $G_R = (G_R(t); t \in \mathbb{T})$ *such that* RKHS(R) = H *a.e.* (λ). *Then* s *is an admissible translate of* X *if and only if* $s \in H$, *and* s *is a singular translate of* X *if and only if* $s \notin H$.

<u>Proof</u>: If $s \in H$, then s is an admissible translate of a.e. G_R (λ), and hence of X.

Now assume $s \notin H$. Let RKHS(R) = H for all $R \in R\setminus\Lambda$, $\lambda(\Lambda) = 0$, and fix $R_0 \in R\setminus\Lambda$. Then for each $n \in \mathbb{N}$, there exist N_n, $a_{n,1}, \ldots, a_{n,N_n}$, $t_{n,1}, \ldots, t_{n,N_n}$, such that

$$\frac{\left|\sum_{k=1}^{N_n} a_{n,k} s(t_{n,k})\right|^2}{E\left|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_{n,k})\right|^2} \geq n.$$

Since for every $R \in \mathcal{R}\backslash\Lambda$, RKHS(R) = H, there exists $0 < c_R < \infty$ such that

$$E\left|\sum_{k=1}^{N_n} a_{n,k} G_R(t_{n,k})\right|^2 \leq c_R E\left|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_{n,k})\right|^2.$$

As in Proposition 2.4, let $s_n = \Sigma_{k=1}^{N_n} a_{n,k} s(t_{n,k})$, (and WLOG assume $s_n > 0$) and $Y_n(x) = \Sigma_{k=1}^{N_n} a_{n,k} x(t_{n,k})$, $x \in \overline{X}^{\mathbb{T}}$, so that

$$\begin{aligned} \mu_{G_R}(Y_n \geq s_n/2) &\leq 2^2 E\left|\sum_{k=1}^{N_n} a_{n,k} G_R(t_n)\right|^2 / s_n^2 \\ &\leq 4c_R E\left|\sum_{k=1}^{N_n} a_{n,k} G_{R_0}(t_n)\right|^2 / s_n^2 \\ &< 4c_R/n \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \mu_{s+G_R}(Y_n \geq s_n/2) &\geq 1 - \mu_{G_R}(|Y_n| > s_n/2) \\ &\geq 1 - 4c_R/n \to 1 \quad \text{as } n \to \infty. \end{aligned}$$

Hence by the dominated convergence theorem

$$\mu_X(Y_n \geq s_n/2) = \int_{\mathcal{R}} \mu_{G_R}(Y_n \geq s_n/2)\lambda(dR) \to 0 \quad \text{as } n \to \infty,$$

and

$$\mu_{s+X}(Y_n \geq s_n/2) = \int_{\mathcal{R}} \mu_{s+G_R}(Y_n \geq s_n/2)\lambda(dR) \to 1 \quad \text{as } n \to \infty.$$

This implies $\mu_X \perp \mu_{s+X}$. Hence every $s \notin H$ is a singular translate of X, and the proof of the dichotomy is complete. $\square$

The assumptions of Proposition 2.5 are satisfied when X is sub-Gaussian, i.e. X is the mixture of the mutually singular Gaussian process $a^{1/2}G$, $a > 0$, which have identical RKHS; or in the more general case where X is the mixture of Gaussian processes with random covariance function of the form $\Sigma_{n=1}^{N} A_n R_n(t,s)$, where the R_n's are fixed (nonrandom) covariances such that $R_n - c_{nm}R_m$ is nonnegative definite for all $n,m = 1,\ldots,N$, and some $0 < c_{nm} < \infty$, and the positive random variables $A_1,\ldots,A_N$, are jointly $\alpha/2$-stable.

The usefulness of these general remarks is limited by the fact that the only SαS mixtures of Gaussian processes, which are currently known explicitly, are the sub-Gaussian processes, and the more general finite sums $\Sigma_{n=1}^{N} A_n^{1/2} G_n$, where $(A_1,\ldots,A_N)$ is positive $\alpha/2$-stable and independent of the mutually independent Gaussian processes $G_1,\ldots,G_N$.

Further examples where the set of admissible translates is trivial or a proper subset of the function space $\mathbb{F}$ are presented in the next section. It should finally be recalled that the set of admissible translates of a SαS process is always a linear space, even if it ~~is~~ not the entire function space $\mathbb{F}$ [24, Corollary 5.1]. However, as will be seen in the next section, the restriction of $\|\cdot\|_{\mathbb{F}}$ to the set of admissible translates may not be the most natural way to define a topology on it. Also, from the linear structure we have that $\mu_{s+X} \ll \mu_X \Rightarrow \mu_X \ll \mu_{s+X}$ (see e.g. [22]) so that for every admissible translate s, μ_{s+X} and μ_X are equivalent.

3. Processes with invertible spectral representation

In this section we present some general results on the admissible translates of certain SαS processes with invertible spectral representation.

Let $X = (X(t); t \in \mathbb{T})$ be an SαS stochastic processes with spectral representation as in (1.2). It follows from the continuity of the stochastic integral map $f \to \int f dZ$ and (1.1) that the representing functions $\{f(t,\cdot); t \in \mathbb{T}\}$ are linearly dense in $L_\alpha(m)$, i.e. that $L(f) = L_\alpha(m)$, where $L(f)$ is the completion of $\ell(f) = sp\{f(t,\cdot); t \in \mathbb{T}\}$ in $L_\alpha(m)$, if and only if $L(X) = L(Z)$. Processes satisfying this condition will be said to have an *invertible spectral representation* or more simply to be *invertible*.

Every Gaussian process is invertible [4 , construction in Theorem 2]. This is not generally true for non-Gaussian SαS processes as can be seen from the fact that the linear space of a sub-Gaussian process does not contain (nontrivial) independent random variables [8 , Lemma 2.1]. Necessary and sufficient conditions for a general SαS process to have an invertible spectral representation are given in [5 , Theorems 5.1 and 5.5]. A stronger form of invertibility for a nonanticipating SαS moving average is considered in [8, Lemma 3.1]. SαS processes with invertible spectral representation in $L_2([0,1], Leb)$, i.e. $L_2 - \overline{sp}\{f(t,\cdot); t \in [0,1]\} = L_2([0,1], Leb)$, are considered in [24]; clearly such a process has also invertible spectral representation in $L_\alpha([0,1], Leb)$. Examples of invertible SαS processes will be presented in the sequel.

For invertible processes the problem of finding their admissible translates can be reduced to finding the admissible translates of the independently scattered measure Z, which we now consider first.

The next proposition is essentially based on [12, Theorem 7.3]. It extends to independently scattered SαS measures with non-atomic control measure the result in [3, 24] on admissible translates of independent increments processes in $[0,T]$ which are stochastically continuous and have no Gaussian component. It establishes a dichotomy for the translates of a general independently scattered SαS random measure and it characterizes its admissible translates as those of its atomic component.

The following notation will be used in Proposition 3.1. Recall that if a σ-finite measure space $(\mathbb{I},\sigma(\mathcal{I}),m)$ is such that $\sigma(\mathcal{I})$ contains all single point sets, e.g. $\mathbb{I}$ is a Polish space, $\sigma(\mathcal{I})$ its Borel sets, and $\mathcal{I}$ the δ-ring of Borel sets with finite m-measure, then $m = m_a + m_d$ where m_a is purely atomic and m_d is diffuse (non-atomic) [15], and the set of atoms is at most countable, say $A = \{a_n; n \in \{1,2,\ldots,N\} \cap \mathbb{N}\}$, N the number of atoms. Thus if $Z = (Z(B); B \in \mathcal{I})$ is an independently scattered SαS random measure with control measure m, it can be expressed as $Z = Z_a + Z_d$, where Z_a and Z_d are independent SαS independently scattered random measures defined for all $B \in \mathcal{I}$ by $Z_a(B) = Z(A \cap B)$ and $Z_d(B) = Z(A^c \cap B)$, and have control measures m_a and m_d respectively. The atomic component has a series expansion

$$Z_a(B) = \sum_{n=1}^{N} 1_B(a_n) Z(\{a_n\})$$

which can be normalized by using the iid standard SαS random variables $Z_n \triangleq Z(\{a_n\}) m^{-1/\alpha}(\{a_n\})$ with $E\{\exp(iuZ_n)\} = \exp(-|u|^\alpha)$, as follows:

$$Z_a(B) = \sum_{n=1}^{N} 1_B(a_n) m^{1/\alpha}(\{a_n\}) Z_n.$$

Proposition 3.1. Let $Z = (Z(B); B \in I)$ *be an independently scattered* $S\alpha S$ *random measure with* $0 < \alpha < 2$ *and control measure* $m = m_a + m_d$, *and let* $S = (S(B); B \in I)$ *be a set function. Then the following are equivalent:*

i) S *is an admissible translate of* Z,

ii) S *is an admissible translate of* Z_a,

iii) S *is concentrated on* A, *i.e.*

$$S(B) = \sum_{n=1}^{N} S(\{a_n\}) 1_B(a_n),$$

and

$$\sum_{n=1}^{N} |S(\{a_n\})|^2 / m^{2/\alpha}(\{a_n\}) < \infty.$$

Furthermore a translate which is not admissible is singular.

<u>Proof</u>: Let ζ_a and ζ_d be the stochastic processes defined on the probability space $(\overline{X}^I, C(\overline{X}^I), \mu_Z)$ with parameter set I by

$$\zeta_a(B,x) = x(A \cap B), \quad \zeta_d(B,x) = x(A^c \cap B), \quad x \in X^I, \quad B \in I.$$

Clearly

(3.1) $\quad \zeta_a(B, Z(\cdot,\omega)) = Z(A \cap B, \omega) = Z_a(B,\omega), \quad \zeta_d(B, Z(\cdot,\omega)) = Z(A^c \cap B, \omega) = Z_d(B,\omega)$, a.s.,

so that ζ_a and ζ_d are independently scattered $S\alpha S$ random measures with control measures m_a and m_d respectively. Let ζ_a and ζ_d also denote the corresponding linear maps $x \to \zeta_a(\cdot,x)$ and $x \to \zeta_d(\cdot,x)$ from $\overline{X}^I$ into $\overline{X}^I$.

i $\Rightarrow$ ii. Suppose $\mu_{S+Z} \ll \mu_Z$. Hence by Proposition 2.4, $S \in \mathbb{F}$ and by definition of $\mathbb{F}$ the map $F: L(Z) \to X$ defined by $F(\Sigma_{k=1}^n a_k Z(A_k)) = \Sigma_{k=1}^n a_k S(A_k)$ is a well defined linear function, so that S is a signed measure on I. Furthermore since $|S(B)| \le C_{p,\alpha} \|S\|_{\mathbb{F}} \|Z(B)\|_\alpha = C_{p,\alpha}\|S\|_{\mathbb{F}}[m(B)]^{1/\alpha}$, S is absolutely continuous with respect to m, i.e. $S(B) = \int_B \bar{z}dm$ for some z locally in $L_1(m): z1_B \in L_1(B)$ for all $B \in I$.

It follows that $\mu_{S+Z}\zeta_d^{-1} \ll \mu_Z\zeta_d^{-1}$ or equivalently $\zeta_d(\cdot,S)$ is an admissible translate of the process ζ_d, since ζ_d is linear. Now

$$\zeta_d(B,S) = S(A^c \cap B) = \int_{A^c\cap B} \bar{z}dm = \int_B \bar{z}dm_d \triangleq S_d(B).$$

Since m_d is nonatomic it follows from a well known result [15 , p. 238] that we can find measurable partitions $\{B_{j,k}(B); k=1,2,\ldots,K_j\}$, $j=1,2,\ldots$, of B for which

$$(3.2) \qquad \max_{1\le k\le K_j} m_d(B_{j,k}(B)) \to 0 \quad \text{as } j\to\infty.$$

For notational simplicity we will omit in the following the dependence on B. It follows that the triangular system of rowwise independent random variables $\{\zeta_d(B_{j,k}); k=1,2,\ldots,K_j,\ j=1,2,\ldots\}$ is infinitesimal, i.e. for every $\varepsilon>0$,

$$\max_{1\le k\le K_j} P(|\zeta_d(B_{j,k})| \ge \varepsilon) \to 0 \quad \text{as } j\to\infty.$$

Hence, since for every j, $\zeta_d(B) = \Sigma_{k=1}^{K_j}\zeta_d(B_{j,k})$, we have from the central limit theorem for triangular arrays and the fact that ζ_d has no Gaussian component that

$$\liminf_{\varepsilon\to 0}\ \liminf_{j\to\infty}\ \mathrm{Var}\{\sum_{k=1}^{K_j}\zeta_d(B_{j,k})1_{(-\varepsilon,\varepsilon)}(|\zeta_d(B_{j,k})|)\} = 0$$

(see e.g. [1 , Theorem 4.7]. Thus by Chebyshev's inequality

(3.3) $$\sum_{k=1}^{K_j} \zeta_d(B_{j,k})1_{(-\varepsilon,\varepsilon)}(|\zeta_d(B_{j,k})|) \to 0$$

in probability (in $L_p(P), p \in (0,\infty)$) as $j \to \infty$ and $\varepsilon \to 0$.

On the other hand, if $S_d(B) = \int_B \bar{z}dm_d$ and $m_d(B_{j,k}) \to 0$ as $j \to \infty$ then $S_d(B_{j,k}) \to 0$ as $j \to \infty$, and hence for j large

$$S_d(B) = \sum_{k=1}^{K_j} S_d(B_{j,k}) = \sum_{k=1}^{K_j} S_d(B_{j,k})1_{(-\varepsilon,\varepsilon)}(|S_d(B_{j,k})|).$$

Similarly

(3.4) $$\sum_{k=1}^{K_j} [S_d(B_{j,k}) + \zeta_d(B_{j,k})]1_{(-\varepsilon,\varepsilon)}(|S_d(B_{j,k}) + \zeta_d(B_{j,k})|) \to S_d(B)$$

in probability as $j \to \infty$ and $\varepsilon \to 0$.

Define for $B \in I$ the map $\phi(B,\cdot):\bar{X}^I \to \bar{X}^I$ by

(3.5) $$\phi(B,x) = \liminf_{\varepsilon \to 0} \liminf_{j \to \infty} \sum_{k=1}^{K_j} x(B_{j,k})1_{(-\varepsilon,\varepsilon)}(|x(B_{j,k})|).$$

Suppose S_d is not identically zero. Then there exists $B \in I$ such that $S_d(B) \neq 0$. It follows from (3.3) and (3.4) that

$$\phi(B,\zeta_d(\cdot,x)) = 0, \quad \phi(B,S_d + \zeta_d(\cdot,x)) = S_d(B) \quad \text{a.s.}$$

Thus $\mu_{S+Z}\phi^{-1}(B,\cdot) \perp \mu_Z\phi^{-1}(B,\cdot)$ and hence $\mu_{S+Z} \perp \mu_Z$ which is a contradiction. Therefore $S_d(B) = \int_B \bar{z}dm_d = 0$ for all $B \in I$, i.e. $z = 0$ a.e. m_d, so that

(3.6) $$S(B) = \int_B \bar{z}dm_a = \sum_{n=1}^{N} \bar{z}(a_n)1_B(a_n)\, m(\{a_n\}).$$

Reasoning as before we have $\mu_{S+Z}\zeta_a^{-1} \ll \mu_Z\zeta_a^{-1}$, i.e. $\zeta_a(\cdot,S)$ is an admissible translate of ζ_a (or Z_a), and by (3.6)

$$\zeta_a(B,S) = S(A\cap B) = \int_{A\cap B} \bar{z}dm = \int_B \bar{z}dm_a = S(B)$$

i.e. $S = \zeta_a(\cdot,S)$ is an admissible translate of Z_a.

<u>ii ⇒ i</u>. Suppose S is an admissible translate of Z_a. Since $Z = Z_a + Z_d$ and Z_a and Z_d are independent we have $\mu_Z = \mu_{Z_a} * \mu_{Z_d}$. Then $\mu_{S+Z_a} \ll \mu_{Z_a}$ implies $\mu_{S+Z} \ll \mu_Z$. Indeed $0 = \mu_Z(B) = \int_{\bar{X}^I} \mu_{Z_a}(B-x)\mu_{Z_d}(dx)$ implies $\mu_{Z_a}(B-x) = 0$ a.e. μ_{Z_d}, hence $0 = \mu_{S+Z_a}(B-x) = \mu_{Z_a}(B-S-x)$ a.e. μ_{Z_d} and thus

$$\mu_{S+Z}(B) = \mu_Z(B-S) = \int_{\bar{X}^I} \mu_{Z_a}(B-S-x)\mu_{Z_d}(dx) = 0.$$

<u>ii ⇒ iii</u>. Because $S \in \mathbb{F}$, S is absolutely continuous with respect to m_d, $S(B) = \Sigma_{n=1}^N S\{a_n\}1_B(a_n)$. Let $\psi:\bar{X}^I \to \bar{X}^N$, where $N = \{1,\dots,N\}$ if $N < \infty$ and $N = \mathbb{N}$ otherwise, be defined by

$$[\psi(x)](n) = \psi(n,x) = \zeta_a(\{a_n\},x)/m^{1/\alpha}(\{a_n\}).$$

Thus by (3.1), $\psi(n,\cdot)$, $n \in N$, are standard SαS iid random variables,

$$\psi(n,S) = \zeta_a(\{a_n\},S)/m^{1/\alpha}(\{a_n\}) = S(\{a_n\})/m^{1/\alpha}(\{a_n\})$$

and

$$\psi(n,S+x) = \psi(n,S) + \psi(n,x) = S(\{a_n\})/m^{1/\alpha}(\{a_n\}) + \psi(n,x).$$

Now $\mu_{S+Z_a} \ll \mu_{Z_a}$ implies $\mu_{S+Z_a}\psi^{-1} \ll \mu_{Z_a}\psi^{-1}$, i.e.

$(S(a_n)/m^{1/\alpha}(\{a_n\});n \in \mathbb{N})$ is an admissible translate of the random element $(\psi(n,\cdot);n \in \mathbb{N})$ defined on the probability space $(\mathbb{R}^T, \mathcal{C}(\mathbb{R}^T), \mu_Z)$. It follows from [20] if $N = \infty$ and trivially if $N < \infty$ that $\Sigma_{n=1}^{N} S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) < \infty$.

iii $\Rightarrow$ ii. Conversely, if $\Sigma_{n=1}^{N} S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) < \infty$ it follows from [20] and the fact that stable densities have finite Fisher information [10] that $(S(\{a_n\})/m^{1/\alpha}(\{a_n\});n \in \mathbb{N})$ is an admissible translate of $(\psi(n,\cdot);n \in \mathbb{N})$ (the result is trivial if $N < \infty$). Therefore

$$\sum_{n=1}^{N} S(\{a_n\}) 1_B(\{a_n\}) = S(B)$$

is an admissible translate of the process

$$\sum_{n=1}^{N} 1_B(\{a_n\}) m^{1/\alpha}(\{a_n\}) \psi(n,x) = \sum_{n=1}^{N} 1_B(a_n) \zeta(\{a_n\},x) = \zeta_a(B,x)$$

and hence of Z_a.

To prove that a translate S which is not admissible is singular it suffices to consider such a translate in $\mathbb{F}$, i.e. from the proof i $\Rightarrow$ ii, $S(B) = \int_B \bar{z} dm$. If $m_d(|z| > 0) > 0$ then $\mu_{S+Z} \perp \mu_Z$. Thus assume $S(B) = \int_B \bar{z} dm_a = \Sigma_{n=1}^{N} S(\{a_n\}) 1_B(a_n)$. Since it is not admissible, by (iii) $N = \infty$ and $\Sigma S^2(\{a_n\})/m^{2/\alpha}(\{a_n\}) = \infty$. Hence from [20] $(S\{a_n\})/m^{1/\alpha}(\{a_n\});n \in \mathbb{N})$ is a singular translate of $(\psi(n,\cdot);n \in \mathbb{N})$, i.e. $\mu_{S+Z}\psi^{-1} \perp \mu_Z\psi^{-1}$ which implies $\mu_{S+Z} \perp \mu_Z$. □

It follows that the admissible translates of an SαS independently scattered random measure are quite different in the Gaussian and non-Gaussian cases. Indeed, for Z Gaussian ($\alpha = 2$) every element in its function space (i.e. its RKHS)

$$\mathbb{F}_2 = \{S; S(B) = \int_B \bar{z}\,dm,\ \bar{z} \in L_2(m)\}$$

$$= \{S; S \text{ signed measure on } \sigma(I),\ S \ll m,\ \frac{ds}{dm} \in L_2(m)\}$$

(see e.g. [9]) is an admissible translate, while, e.g. for Z non-Gaussian with $1 < \alpha < 2$ its only admissible translates are $S(B) = \int_B \bar{z}\,dm$, $z \in L_{\alpha^*}(m)$, with $z = 0$ a.e. m_d, and $\Sigma_{n=1}^N |S(\{a_n\})|^2/m^{2/\alpha}(\{a_n\}) < \infty$. Hence for $1 < \alpha < 2$ the set of admissible translates is a proper subset of the function space $\mathbb{F}_\alpha$ which is given by

$$\mathbb{F}_\alpha = \{S : S \text{ signed measure on } \sigma(I),\ S \ll m,\ \frac{ds}{dm} \in L_{\alpha^*}(m)\}.$$

In particular, while a diffuse Gaussian random measure has a rich class of admissible translates, a diffuse non-Gaussian SαS random measure has no admissible translate whatever. On the other hand, if m (or Z) is atomic ($m_d = 0$), the condition in Proposition 3.1 (iii) extends the Gaussian condition. Indeed if $\alpha = 2$ and $S(B) = \int_B \frac{ds}{dm} dm = \Sigma_{n=1}^N \frac{ds}{dm}(a_n) m(\{a_n\})$, then $\Sigma |S(\{a_n\})|^2/m(\{a_n\})^{2/\alpha} < \infty$ is equivalent to $\frac{ds}{dm} \in L_2(m)$.

The results of Proposition 3.1 can now be used to obtain a dichotomy for the translates of an invertible SαS process, and to characterize its admissible translates as those of its atomic component. In order to state the result for a SαS process X with spectral representation $X(t) = \int_I f(t,u) Z(du)$ and control measure m, we introduce the independent SαS diffuse and atomic component processes of X:

$$X_d(t) = \int_{A^c} f(t,u)Z(du) = \int_{\mathbb{T}} f(t,u)Z_d(t),$$

$$X_a(t) = X(t) - X_d(t) = \int_A f(t,u)Z(du) = \int_{\mathbb{T}} f(t,u)Z_a(du).$$

The atomic component X_a has a series expansion

$$X_a(t) = \sum_{n=1}^{N} f(t,a_n)Z(\{a_n\}) ,$$

which can be normalized by putting

$$Z_n = Z(\{a_n\})/m^{1/\alpha}(\{a_n\}), \quad f_n(t) = f(t,a_n)m^{1/\alpha}(\{a_n\}),$$

so that the Z_n's are standard SαS iid random variables, for all $t \in \mathbb{T}$, $\Sigma_{n=1}^{N} |f_n(t)|^{\alpha} < \infty$, and

$$X_a(t) = \sum_{n=1}^{N} f_n(t)Z_n.$$

Proposition 3.2. *Let* $X = (X(t); t \in \mathbb{T})$ *be a* SαS *process with* $0 < \alpha < 2$, *invertible spectral representation* $X(t) = \int_{\mathbb{T}} f(t,u)Z(du)$ *and control measure* m, *and let* $s = (s(t); t \in \mathbb{T})$ *be a function on* $\mathbb{T}$. *Then the following are equivalent:*

i) s *is an admissible translate of* X,

ii) s *is an admissible translate of* X_a,

iii) $s(t) = \Sigma_{n=1}^{N} s'_n f(t,a_n)$ *with* $\Sigma_{n=1}^{N} |s'_n|^2/m^{2/\alpha}(\{a_n\}) < \infty$

$$= \Sigma_{n=1}^{N} s_n f_n(t) \text{ with } \Sigma_{n=1}^{N} |s_n|^2 < \infty.$$

Furthermore a translate which is not admissible is singular.

Proof: i. Since $1_B \in L_\alpha(m) = L(f)$, for any $B \in I$, there exist $\Phi_n(B,\cdot) \in sp\{f(t,\cdot);t \in \mathbb{T}\}$, $n=1,2,\ldots$, i.e.

$\Phi_n(B,\cdot) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) f(t_{n,k}(B),\cdot)$ such that

$\Phi_n(B,\cdot) \to 1_B$ in $L_\alpha(m)$ as $n \to \infty$. Define

$$\Phi_n(B,x) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) x(t_{n,k}(B)), \quad x \in \bar{X}^{\mathbb{T}}.$$

Thus

$$\Phi_n(B,X(\cdot,\omega)) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) X(t_{n,k}(B),\omega) \tag{3.7}$$

$$= \int_{\mathbb{I}} \phi_n(B,u) Z(du,\omega) \to \int_{\mathbb{I}} 1_B(u) Z(du,\omega) = Z(B,\omega)$$

in $L_p(P)$ (hence in probability) as $n \to \infty$. Thus $(\Phi_n(B,\cdot);n \in \mathbb{N})$ converges in μ_X-measure. Let $(\Phi_{n_k}(B,\cdot);k \in \mathbb{N})$ be a subsequence converging a.e. μ_X and define

$$\tilde{Z}(B) = \tilde{Z}(B,\cdot) = \liminf_{k\to\infty} \Phi_{n_k}(B,\cdot) 1_{\{x;\Phi_{n_k}(B,x) \text{ converges}\}}(\cdot).$$

$\tilde{Z}(B,\cdot)$ is a well defined C-measurable function on $\bar{X}^{\mathbb{T}}$ for each $B \in I$. Hence $\tilde{Z} = (\tilde{Z}(B);B \in I)$ is a stochastic process on the probability space $(\bar{X}^{\mathbb{T}},C,\mu_X)$, and from (3.7), $\tilde{Z}(B,X(\cdot,\omega)) = Z(B,\omega)$ a.s., so that $\tilde{Z}$ is equal in law to Z, i.e. $\tilde{Z}$ is an independently scattered SαS random measure with control measure m.

i $\Rightarrow$ ii. Let s be an admissible translate of X. From Proposition 2.4, $s \in \mathbb{F}$, i.e.

$$\left|\sum_{k=1}^{n} a_k s(t_k)\right| \le \|s\|_{\mathbb{F}} \left\|\sum_{k=1}^{n} a_k X(t_k)\right\|_{L_p(P)}, \quad p \in (0,\alpha).$$

Hence as in Proposition 3.1, $F(\sum_{k=1}^{n} a_k X(t_k)) = \sum_{k=1}^{n} a_k s(t_k)$ is a well defined continuous linear functional on $L(X)$ and $s(t) = F(X(t))$. Thus

$$\tilde{Z}_n(B,s) = \sum_{k=1}^{N_n(B)} a_{n,k}(B) s(t_{n,k}(B))$$

$$= F\Big(\sum_{k=1}^{N_n(B)} a_{n,k}(B) X(t_{n,k}(B))\Big) \to F(Z(B)).$$

Hence for all $B \in I$,

$$(3.9) \qquad \tilde{Z}(B,s) = F(Z(B))$$

and

$$(3.10) \qquad \tilde{Z}(B,s+x) = \tilde{Z}(B,s) + \tilde{Z}(B,x).$$

Now if $\tilde{Z}_d(B,\cdot) = \tilde{Z}(A^c \cap B,\cdot)$, then $\tilde{Z}_d = (\tilde{Z}_d(B,\cdot); B \in I)$ is an independently scattered SαS random measure with control measure m_d and by (3.10) it has $\tilde{Z}_c(\cdot,s)$ as an admissible translate. But m_d is non-atomic, thus by Proposition 3.1, $\tilde{Z}_d(\cdot,s) = 0$, i.e. for all $B \in I$,

$$0 = \tilde{Z}_d(B,s) = \tilde{Z}(A^c \cap B,s) = F(Z(A^c \cap B)) = F(Z_d(B)),$$

and hence

$$s(t) = F(X(t)) = F(X_a(t) + X_d(t)) = F(X_a(t))$$

(since X_d is obtained by a linear operation on Z_d which implies $F(X_d(t)) = 0$). Therefore

$$(3.11)\qquad s(t) = F(X_a(t)) = F\left(\sum_{n=1}^{N} f_n(t) Z_n\right)$$

$$= \sum_{n=1}^{N} f_n(t) F(Z_n) = \sum_{n=1}^{N} f_n(t) s_n$$

$$= \sum_{n=1}^{N} f_n(t, a_n) s_n'$$

where $s_n = F(Z_n)$ and $s_n' = m^{1/\alpha}(\{a_n\}) s_n$. On the other hand $\tilde{X}_a = (\tilde{X}_a(t,x) = \sum_{n=1}^{N} f(t,a_n)\tilde{Z}(\{a_n\},x); t \in \mathbb{T})$ has distribution μ_{X_a} and by the linearity of the map $x \to \tilde{X}_a(\cdot,x)$, the function $\tilde{X}_a(\cdot,s)$ is an admissible translate of $\tilde{X}$ and hence of X_a. But

$$\tilde{X}_a(t,s) = \sum_{n=1}^{N} f(t,a_n)\tilde{Z}(\{a_n\},s)$$

$$= \sum_{n=1}^{N} f(t,a_n) F(Z(\{a_n\}))$$

$$= \sum_{n=1}^{N} f(t,a_n) s_n' = \sum_{n=1}^{N} f_n(t) s_n = s(t),$$

i.e. s is an admissible translate of X_a.

<u>ii $\Rightarrow$ i</u>. The proof is identical to that in Proposition 3.1.

<u>ii $\Leftrightarrow$ iii</u>. The proof is as in Proposition 3.1, with $\psi(n,x) = \tilde{Z}(\{a_n\},x)/m^{1/\alpha}(\{a_n\})$, so that by (3.9),

$$\psi(n,s) = \tilde{Z}(\{a_n\},s)/m^{1/\alpha}(\{a_n\})$$

$$(3.12)\qquad = F(Z(\{a_n\}))/m^{1/\alpha}(\{a_n\}) = s_n'/m^{1/\alpha}(\{a_n\})$$

$$= s_n.$$

To prove that a translate which is not admissible is singular, it suffices to consider $s \in \mathbb{F}(X)$, i.e., $s(t) = F(X(t))$, as by Proposition 2.4, $s \notin \mathbb{F}$ implies singularity. Suppose $F(X_d(t)) \neq 0$. Then there exists $B \in I$ such that $F(Z_d(B)) \neq 0$ and by (3.9),

$$\tilde{Z}_d(B,s) = \tilde{Z}(A^c \cap B,s) = F(Z(A^c \cap B)) = F(Z_d(B)) \neq 0.$$

It follows from Proposition 3.1 that $\mu_{s+X}\tilde{Z}_d^{-1} \perp \mu_X \tilde{Z}_d^{-1}$ and hence $\mu_{s+X} \perp \mu_X$. Therefore $s(t) = F(X_a(t)) = \Sigma_{n=1}^N f_n(t) s_n$ and as in the proof of Proposition 3.1, $\Sigma_{n=1}^N |s_n|^2 = \infty$ implies $\mu_{s+X} \perp \mu_X$. $\square$

It follows from Proposition 3.2 that for an invertible SαS process with nonatomic control measure every non-zero translate is singular. In particular, this contains Corollary 10.1 of [24]. Applied to SαS processes with purely atomic control measure, Proposition 3.2 is a stochastic process version of a result proved in [22, Theorem 4] for SαS measures with discrete spectral measures on separable Banach spaces. The proposition completes the result in [22] providing a dichotomy for the problem of admissible translates.

Proposition 3.2 also provides examples where the set of admissible translates is a non-trivial proper subset of the function space $\mathbb{F}$ of the process X. E.g. if $X(t) = \Sigma_{n=1} f_n(t) Z_n$, $t \in \mathbb{T}$, where $Z_1, Z_2, \ldots$ are iid standard SαS random variables with $1 < \alpha \leq 2$ and $\ell_\alpha - \overline{sp}\{(f_n(t); n \in \mathbb{N}); t \in \mathbb{T}\} = \ell_\alpha$, then

$$\mathbb{F}_\alpha = \{s : s(t) = \sum_{n=1}^{\infty} s_n f_n(t), \ \sum_{n=1}^{\infty} |s_n|^{\alpha^*} < \infty\},$$

while the set of admissible translates is the infinite dimensional subspace (since $\alpha^* \geq 2$) of $\mathbb{F}_\alpha$ for which $\sum_{n=1}^{\infty}|s_n|^2 < \infty$; hence we have equality only if $\alpha = 2$, and proper inclusion if $1 < \alpha < 2$. There is a natural identification between the set of admissible translates, which is always a linear space, and the Hilbert space ℓ_2, namely $(s_n; n \in \mathbb{N}) \to s(t) = \sum_{n=1}^{\infty} s_n f_n(t)$. This map is invertible with inverse map given by the transformation ψ defined in the proof of Proposition 3.2 restricted to the set of admissible translates of X (cf. (3.12)). Thus for every $\alpha \in (0,2)$, the linear space of admissible translates can be given a Hilbert space structure by defining the inner product

$$<s_1, s_2> = <(s_{1,n}),(s_{2,n})>_{\ell_2} = \sum_{n=1}^{\infty} s_{1,n} s_{2,n}$$

where $s_i(t) = \sum_{n=1}^{\infty} s_{i,n} f_n(t)$, $i = 1,2$ Note that in this case when $1 < \alpha < 2$, $\|s\|_{\mathbb{F}} = (\sum|s_n|^{\alpha^*})^{1/\alpha^*}$ and hence $\|\cdot\|_{\mathbb{F}}$ is not a natural norm on the linear space of admissible translates, in contrast with the case of Gaussian ($\alpha = 2$) and α-sub-Gaussian processes with $1 < \alpha < 2$.

Important examples of SαS processes with invertible representation are presented in the following.

<u>Harmonizable SαS processes (and sequences)</u>.

Let $X = (X(t); t \in \mathbb{T})$, $\mathbb{T} = \mathbb{R}^d$ or $\mathbb{Z}^d$, $d \in \mathbb{N}$, be an SαS harmonizable process, i.e., X has the representation

$$X(t) = \int_{\mathbb{I}} e^{i<t,u>} Z(du), \quad t \in \mathbb{T},$$

where $\mathbb{I} = \mathbb{R}^d$ or $[-\pi, \pi]^d$ for $\mathbb{T} = \mathbb{R}^d$ and $\mathbb{Z}^d$ respectively and Z is an SαS independently scattered complex random measure with finite control measure m, referred to as the spectral measure of the harmonizable process X. If the spectral measure m is nonatomic and $0<\alpha<2$ then it follows from Proposition 3.2 that X has no nontrivial admissible translate. When the stable distribution of Z is radially symmetric, i.e., when X is stationary, this result exhibits a different behavior compared with the stationary Gaussian processes $\alpha=2$, whose admissible translates are precisely the functions $s(t)=\int_{\mathbb{I}} e^{i<t,u>}\bar{z}(u)m(du)$, $z\in L_2(m)$. In contrast, if m is purely discrete, i.e. X has a Fourier series representation

$$X(t) = \sum_{n=1}^{N} b_n e^{i<c_n,t>} Z_n, \quad N\le\infty,$$

with Z_n's iid standard SαS random variables and $\Sigma_{n=1}^{\infty}|b_n|^{\alpha}<\infty$, the set of admissible translates is

$$\{s: s(t) = \sum_{n=1}^{N} s_n e^{i<c_n,t>} ; \sum_{n=1}^{N} |s_n/b_n|^2 <\infty\},$$

and depends on α, $0<\alpha\le 2$, only via the sequence $(b_n; n\in\mathbb{N})\in\ell_{\alpha}$. In other words for fixed $(b_n; n\in\mathbb{N})\in\ell_{\beta}$, $1\le\beta\le 2$, define $X_{\alpha}(t) = \Sigma_{n=1}^{\infty} b_n \exp\{i<c_n,t>\}Z_{n,\alpha}$, where the $Z_{n,\alpha}$'s are standard iid SαS with $\beta\le\alpha\le 2$ for $1<\beta\le 2$ and $1<\alpha\le 2$ for $\beta=1$. Then all these processes X_{α} have the same set of admissible translates.

<u>Continuous time SαS moving averages</u>.

Another class of stationary SαS processes is the class of real moving averages,

$$X(t) = \int_{\mathbb{R}} f(t-u)Z(du), \quad t\in\mathbb{R},$$

where Z has Lebesgue control measure and $f\in L_{\alpha}(\mathrm{Leb})$. When f vanishes on the negative line, they are called nonanticipating moving averages and they occur as the stationary solutions of n^{th}

order linear stochastic differential equations with constant coefficients driven by stable motion Z.

In the Gaussian case $\alpha = 2$ the admissible translates coincide with the function space (RKHS)

$$\mathbb{F}_2 = \{s : s(t) = \int f(t-u)z(u)du,\ z \in L_2(\mathrm{Leb})\}$$

$$= \{s ; s \in L_2(\mathrm{Leb}),\ \hat{s}/\hat{f} \in L_2(\mathrm{Leb})\},$$

where ^ denotes Fourier transform.

Examples of moving averages with invertible spectral representation and therefore with no admissible translates, can be obtained by taking

i) f continuous and equal to zero on $(-\infty, 0)$ and at infinity [2 , Theorem 2],

ii) $\alpha \in (1,2)$ and f a Fourier transform of some function F in $L_{\alpha^*}(\mathrm{Leb})$ with $F \neq 0$ a.e. [23 , Theorem 75].

Case i) includes nonanticipating moving averages with continuous kernel f, while case ii) contains certain nonanticipating moving averages with discontinuous kernels f, namely the stationary solutions of n^{th} order linear stochastic differential equations with constant coefficients. There f(t) is a linear combination of functions of the type $t^{k-1}e^{-at}1_{[0,\infty)}(t)$ with $k \in \mathbb{N}$ and $a > 0$, which are Fourier transforms of the $L_{\alpha^*}(\mathrm{Leb})$ functions $\Gamma(k)/[2\pi(a+iu)]$. Hence f is the Fourier transform of an $L_{\alpha^*}(\mathrm{Leb})$ function which is $\neq 0$, a.e., so that $\overline{sp}\{f(t-\cdot); t \in \mathbb{R}\} = L_\alpha(\mathrm{Leb})$, i.e. X is invertible. Thus solutions of n^{th} order stochastic differential equations driven by SαS motion have no admissible translate for $1 < \alpha < 2$. This is in sharp contrast with the Gaussian case $\alpha = 2$. E.g., if $n = 1$, $f(t) = e^{-t} 1_{[0,\infty)}(t)$ and the stable Ornstein-Uhlenbeck (OU) process

$$X(t) = \int_{-\infty}^{t} e^{-(t-u)} Z(du)$$

has no admissible translates for $1 < \alpha < 2$ while for $\alpha = 2$ all translates of the form

$$s(t) = \int_{-\infty}^{t} e^{-(t-u)} z(u)du, \ z \in L_2(\text{Leb}),$$

are admissible for the OU process X.

Discrete time SαS processes (SαS sequences) with invertible spectral representation have similar sets of admissible translates in the Gaussian and non-Gaussian stable case. Of course nonadmissible translates are singular.

Independent sequences and partial sums of independent SαS random variables.

The set of admissible translates of a sequence of independent SαS random variables $X = (X_n; n \in \mathbb{N})$ is given by

$$\{s = (s_n; n \in \mathbb{N}); \sum (s_n/\|X_n\|_\alpha)^2 < \infty\}.$$

The admissible translates of a sequence $(Y_n = \sum_{k=1}^n X_k, n \in \mathbb{N})$ of partial sums of independent SαS random variables X_K are

$$\{s = (s_n; n \in \mathbb{N}); \sum_{n=1}^{\infty} (s_n - s_{n-1})^2/\|X_n\|_\alpha^2 < \infty, \ s_0 = 0\}.$$

Mixed auto-regressive moving averages of order (p,q) (ARMA(p,q)).

Let $X = (X_n; n \in \mathbb{N})$ be defined by the difference equation

$$X_n - a_1 X_{n-1} - \dots - a_p X_{n-p} = Z_n + b_1 Z_{n-1} + \dots + b_q Z_{n-q}$$

where $Z = (Z_n; n \in \mathbb{N})$ is a sequence of iid standard SαS random variables. If the polynomials $P(u) = 1 - a_1 u - \dots - a_p u^p$ and $Q(u) = 1 + b_2 u + \dots + b_q u^q$ satisfy the condition $P(u)Q(u) \neq 0$ for all $u \in \mathbb{C}$ with $|u| \le 1$, then the difference equation defining X has a unique stationary solution of the moving average form

$$X_n = \sum_{k=-\infty}^{n} g_{n-k} Z_k$$

and in addition

$$Z_n = X_n - \sum_{j=1}^{\infty} h_j X_{n-j}.$$

The coefficients $\{g_n; n \in \mathbb{N}\}$ and $\{h_n; n \in \mathbb{N}\}$ are uniquely determined by the power series expansions

$$Q(u)/P(u) = \sum_{j=0}^{\infty} g_j u^j \quad \text{and} \quad P(u)/Q(u) = 1 - \sum_{j=1}^{\infty} h_j u^j, \ |u| \le 1,$$

respectively. Thus $L(X) = L(Z)$, i.e. X is invertible, and hence, by Proposition 3.2, $s = (s_n; n \in \mathbb{Z})$ is an admissible translate of X if and only if it is of the form

$$s_n = \sum_{k=-\infty}^{n} g_{n-k} z_k$$

where $\Sigma_{k=-\infty}^{\infty} z_k^2 < \infty$.

We should note the different behavior of moving averages in continuous and in discrete time. A continuous time moving average may have no admissible translates, whereas a discrete time ARMA sequence has a set of admissible translates identical to the Gaussian case. The difference will be in the form of the Radon-Nikodym derivatives.

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